

On the Skitovich-Darmois theorem for \mathbf{a} -adic solenoids

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Abstract. Let X be a compact connected Abelian group. It is well-known that then there exist topological automorphisms α_j, β_j of X and independent random variables ξ_1 and ξ_2 with values in X and distributions μ_1, μ_2 such that the linear forms $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ and $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$ are independent, whereas μ_1 and μ_2 are not represented as convolutions of Gaussian and idempotent distributions. This means that the Skitovich–Darmois theorem fails for such groups. We prove that if we consider three linear forms of three independent random variables taking values in X , where X is an \mathbf{a} -adic solenoid, then the independence of the linear forms implies that at least one of the distributions is idempotent. We describe all such solenoids.

1. Introduction

It is well-known that proofs of many characterization theorems of mathematical statistics are reduced to solving of some functional equations. Consider the classical Skitovich–Darmois theorem that characterizes Gaussian distributions on the real line ([8, ch. 3]): Let $\xi_i, i = 1, 2, \dots, n, n \geq 2$, be independent random variables, and α_j, β_j be nonzero constants. Suppose that the linear forms $L_1 = \alpha_1 \xi_1 + \dots + \alpha_n \xi_n$ and $L_2 = \beta_1 \xi_1 + \dots + \beta_n \xi_n$ are independent. Then all random variables ξ_j are Gaussian.

Let $\hat{\mu}_j(y)$ be the characteristic functions of the distributions of $\xi_j, j = 1, 2, \dots, n$. Taking in the account that $\mathbf{E}[e^{i\xi_j y}] = \hat{\mu}_j(y)$, it is easy to verify that the Skitovich–Darmois theorem is equivalent to the following statement: The solutions of the Skitovich–Darmois equation

$$\prod_{j=1}^n \hat{\mu}_j(\alpha_j u + \beta_j v) = \prod_{j=1}^n \hat{\mu}_j(\alpha_j u) \hat{\mu}_j(\beta_j v), \quad u, v \in \mathbb{R},$$

in the class of the normalized continuous positive definite functions are the characteristic functions of the Gaussian distributions, i.e. $\hat{\mu}_j(y) = \exp\{ia_jy - \sigma_jy^2\}$, $a_j \in \mathbb{R}, \sigma_j \geq 0, y \in \mathbb{R}, j = 1, 2, \dots, n$.

This theorem was generalized to various classes of locally compact Abelian groups (see for example [1]–[5], [9]). In these researches random variables take values in a locally compact Abelian group X , and coefficients of the linear forms are topological automorphisms of X . As in the classical case the characterization problem is reduced to the solving of the Skitovich–Darmois equation in the class of the normalized continuous positive definite functions on the character group of the group X .

In [2] Feldman and Graczyk have shown, that even a weak analogue of the Skitovich–Darmois theorem fails for compact connected Abelian groups. Namely, they proved the following statement: Let X be an arbitrary compact connected Abelian group. Then there exist topological automorphisms $\alpha_j, \beta_j, j = 1, 2$, of X and independent random variables ξ_1, ξ_2 with values in X and having distributions, that are not convolutions of the Gaussian and idempotent distributions, whereas the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent.

The aim of this article is to show that a weak analogue of the Skitovich–Darmois theorem holds for some compact connected Abelian groups if we consider three linear forms of three random variables. Namely, we will construct an \mathbf{a} -adic solenoid $\Sigma_{\mathbf{a}}$ (we give the full description of such solenoids in the Theorem 4.1) for which the independence of three linear forms of three independent random variables with values in $\Sigma_{\mathbf{a}}$ implies that at least one random variable has idempotent distribution.

2. Definitions and notation

Let X be a second countable locally compact Abelian group. Denote by $\text{Aut}(X)$ the group of the topological automorphisms of X . Let k be an integer. Denote by f_k the mapping $f_k : X \rightarrow X$ definite by the equality $f_kx = kx$. Put $X^{(k)} = f_k(X)$.

Let $Y = X^*$ be the character group of X . The value of a character $y \in Y$ at $x \in X$ denote by (x, y) . Let B be a nonempty subset of X . Put

$$A(Y, B) = \{y \in Y : (x, y) = 1, x \in B\}.$$

The set $A(Y, B)$ is called the annihilator of B in Y . The annihilator $A(Y, B)$ is a closed subgroup in Y . For each $\alpha \in \text{Aut}(X)$ definite the mapping $\tilde{\alpha} : Y \rightarrow Y$ by the equality $(\alpha x, y) = (x, \tilde{\alpha}y)$ for all $x \in X, y \in Y$. The mapping $\tilde{\alpha}$ is a topological automorphism of Y . It is called an adjoint of α . The identity automorphism of a group X denote by I .

In the paper we will use standard facts of abstract harmonic analysis (see [11]). Let μ be a distribution on X . The characteristic function of μ is

definite by the formula

$$\hat{\mu}(y) = \int_X (x, y) d\mu(y), y \in Y.$$

Put $F_\mu = \{y \in Y : \hat{\mu}(y) = 1\}$. Then F_μ is a subgroup of Y , and the function $\hat{\mu}(y)$ is F_μ -invariant, i.e. $\hat{\mu}(y + h) = \hat{\mu}(y)$, $y \in Y, h \in F_\mu$.

Denote by E_x the degenerate distribution concentrated at x . Let K be a compact subgroup of X . Denote by m_K the Haar distribution on K . Denote by $I(X)$ the set of shifts of such distributions, i.e. the distributions of the form $m_K * E_x$, where K is a compact subgroup of X , $x \in X$. The distributions of the class $I(X)$ are called idempotent. Note that the characteristic function of m_K is of the form:

$$\hat{m}_K(y) = \begin{cases} 1, & y \in A(Y, K), \\ 0, & y \notin A(Y, K). \end{cases}$$

A distribution μ on the group X is called Gaussian ([12]) if its characteristic function can be represented in the form

$$\hat{\mu}(y) = (x, y) \exp\{-\varphi(y)\}, \quad y \in Y,$$

where $\varphi(y)$ is a continuous nonnegative function satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2(\varphi(u) + \varphi(v)), \quad u, v \in Y.$$

Denote by $\Gamma(X)$ the set of Gaussian distributions on X .

Denote by \mathbb{Z} the infinite cyclic group, by \mathbb{R} the additive group of real numbers, by \mathbb{T} the circle group, by \mathbb{Q} the additive group of rational numbers with the discrete topology, by Δ_α the group of α -adic integers, by $\mathbb{Z}(m)$ the group of residue modulo m .

Let $\alpha = (a_0, a_1, \dots, a_n, \dots)$ be a fixed but arbitrary infinite sequence of natural numbers, where all $a_i > 1$. Consider the group $\mathbb{R} \times \Delta_\alpha$. Let B be a subgroup of $\mathbb{R} \times \Delta_\alpha$ of the form $B = \{(n, n\mathbf{u})\}_{n=-\infty}^\infty$, where $\mathbf{u} = (1, 0, \dots, 0, \dots)$. The factor-group $\Sigma_\alpha = (\mathbb{R} \times \Delta_\alpha)/B$ is called an α -adic solenoid. The group Σ_α is a compact connected Abelian group and has dimension 1. Moreover $\Sigma_\alpha^* \cong H_\alpha$, where

$$H_\alpha = \left\{ \frac{m}{a_0 a_1 \cdots a_n} : n = 0, 1, \dots; m \in \mathbb{Z} \right\},$$

is subgroup of \mathbb{Q} . Denote by \mathcal{P} the set of prime numbers.

3. Lemmas

Let X be a locally compact Abelian group. Put $Y = X^*$, $\tilde{\alpha}_{ij} \in \text{Aut}(Y)$, $i, j = 1, 2, \dots, n$. Let $f_i(y)$ be some functions on Y . Recall that the Skitovich-Darmois equation is an equation of the form:

$$\prod_{i=1}^n f_i \left(\sum_{j=1}^n \tilde{\alpha}_{ij} u_j \right) = \prod_{i=1}^n \prod_{j=1}^n f_i(\tilde{\alpha}_{ij} u_j), \quad u_j \in Y. \quad (1)$$

The proof of the main theorem is reduced to the studying of the solutions of this equation. In order to prove the main result we need some lemmas.

Lemma 3.1. ([10]). *Let X be a second countable locally compact Abelian group, $\xi_i, i = 1, 2, \dots, n$, be independent random variables with values in X , and distributions μ_i . The linear forms $L_j = \sum_{i=1}^n \alpha_{ij} \xi_i, j = 1, 2, \dots, n$, where $\alpha_{ij} \in \text{Aut}(X)$, are independent if and only if the characteristic functions $\hat{\mu}_i(y), i = 1, 2, \dots, n$, satisfy equation (2), which takes the form*

$$\prod_{i=1}^n \hat{\mu}_i \left(\sum_{j=1}^n \tilde{\alpha}_{ij} u_j \right) = \prod_{i=1}^n \prod_{j=1}^n \hat{\mu}_i(\tilde{\alpha}_{ij} u_j), \quad u_j \in Y. \quad (2)$$

Lemma 3.2. ([10]). *Let X be a direct product of groups $\mathbb{Z}(p^{k_p})$, where $k_p \geq 0$, i.e. $X = \mathbf{P}_{p \in \mathcal{P}} \mathbb{Z}(p^{k_p})$. Let $\xi_i, i = 1, 2, \dots, n$, be independent random variables with values in X and distributions μ_i . Then the independence of the linear forms $L_j = \sum_{i=1}^n \alpha_{ij} \xi_i$, where $\alpha_{ij} \in \text{Aut}(X), \alpha_{1j} = \alpha_{i1} = I, i, j = 1, 2, \dots, n$, implies that $\mu_i = E_{x_i} * m_K$, where K is a compact subgroup of $X, x_i \in X, i = 1, 2, \dots, n$.*

Taking into the account that $X = \mathbf{P}_{p \in \mathcal{P}} \mathbb{Z}(p^{k_p})$ if and only if Y is a weak direct product of the groups $\mathbb{Z}(p^{k_p})$, where $k_p \geq 0$, i.e. $Y = \mathbf{P}_{p \in \mathcal{P}}^* \mathbb{Z}(p^{k_p})$, by lemmas 3.1 and 3.2 we obtain

Corollary 3.3. *Let Y be a discrete Abelian group of the form $Y = \mathbf{P}_{p \in \mathcal{P}}^* \mathbb{Z}(p^{k_p})$, where $k_p \geq 0$. Let $\hat{\mu}_i(y), i = 1, 2, \dots, n, n \geq 2$, be the characteristic functions on Y , satisfying equation (2), where $\tilde{\alpha}_{ij} \in \text{Aut}(Y), \tilde{\alpha}_{1j} = \tilde{\alpha}_{i1} = I, i, j = 1, 2, \dots, n$. Then $\hat{\mu}_i(y) = (x_i, y) \hat{m}_K(y), y \in Y$, where K is a compact subgroup of $X, x_i \in X, i = 1, 2, \dots, n$.*

The following lemma states that an analogue of the Skitovich–Darmois theorem for three linear forms of three independent random variables holds on the circle group if we assume that the characteristic functions of the random variables do not vanish.

Lemma 3.4. ([7]) *Assume that $X = \mathbb{T}, \alpha_{ij} \in \text{Aut}(X), i, j = 1, 2, 3$. Let $\xi_i, i = 1, 2, 3$, be independent random variables with values in X and distributions μ_i , such that their characteristic functions do not vanish. Suppose that $L_j = \sum_{i=1}^3 \alpha_{ij} \xi_i, j = 1, 2, 3$, are independent. Then $\mu_i = E_{x_i}, x_i \in X, i = 1, 2, 3$.*

By Lemmas 3.1 and 3.4 we obtain

Corollary 3.5. *Assume that $Y = \mathbb{Z}$. Let $\hat{\mu}_i(y), i = 1, 2, 3, n \geq 2$, be non-vanishing characteristic functions on Y satisfying the equation*

$$\begin{aligned} & \hat{\mu}_1(u_1 + u_2 + u_3) \hat{\mu}_2(u_1 - u_2 - u_3) \hat{\mu}_3(u_1 + u_2 - u_3) = \\ & = \hat{\mu}_1(u_1) \hat{\mu}_1(u_2) \hat{\mu}_1(u_3) \hat{\mu}_2(u_1) \hat{\mu}_2(-u_2) \hat{\mu}_2(-u_3) \hat{\mu}_3(u_1) \hat{\mu}_3(u_2) \hat{\mu}_3(-u_3), \\ & \quad u_i \in Y, i = 1, 2, 3. \end{aligned} \quad (3)$$

Then $\hat{\mu}_i(y) = (x_i, y), x_i \in X, i = 1, 2, 3, y \in Y$.

Lemma 3.6. ([6, Lemma 13.20]) *Let X be a second countable compact Abelian group. Suppose that there exists an automorphism $\delta \in \text{Aut}(X)$ and an element $\tilde{y} \in Y$, such that the following conditions are satisfied:*

- i) $\text{Ker}(I - \tilde{\delta}) = \{0\}$;
- ii) $(I - \tilde{\delta})Y \cap \{0; \pm\tilde{y}, \pm 2\tilde{y}\} = \{0\}$;
- iii) $\tilde{\delta}\tilde{y} \neq -\tilde{y}$.

*Then for all $n \geq 2$ there exist independent identically distributed random variables $\xi_i, i = 1, 2, \dots, n$, with values in X and distribution $\mu \notin I(X) * \Gamma(X)$, such that the linear forms $L_j = \xi_1 + \sum_{i=2}^n \delta_{ij}\xi_i, j = 1, 2, \dots, n$, where $\delta_{ij} = I, i \neq j, \delta_{ii} = \delta$, are independent.*

It is convenient for us to formulate as a lemma the following simple statement.

Lemma 3.7. *Let Y be a second countable discrete Abelian group, H be a subgroup of Y , $f(y)$ be a function on Y of the form*

$$f(y) = \begin{cases} 1, & y \in H; \\ c, & y \notin H, \end{cases} \quad (4)$$

where $0 < c < 1$. Then $f(y)$ is a positive definite function.

Proof. Consider the distribution $\mu = cE_0 + (1 - c)m_G$ on the group X , where $G = A(X, H)$. It is easy to see that $f(y) = \hat{\mu}(y)$. Hence, $f(y)$ is a positive definite function. ■

The following lemma for $n = 2$ was proved in [2].

Lemma 3.8. *Let X be a second countable compact connected Abelian group, such that $f_2 \in \text{Aut}(X)$. Then there exist independent random variables $\xi_i, i = 1, 2, \dots, n$, with values in X and distributions $\mu_i \notin I(X) * \Gamma(X)$, and automorphisms $\alpha_{ij} \in \text{Aut}(X)$, such that the linear forms $L_j = \sum_{i=1}^n \alpha_{ij}\xi_i, j = 1, 2, \dots, n$, are independent.*

Proof. Two cases are possible: 1. $f_p \in \text{Aut}(X)$ for all prime number p ; 2. $f_p \notin \text{Aut}(X)$ for a prime number p .

1. Consider the first case. It is well-known that if X is a compact Abelian group X such that $f_p \in \text{Aut}(X)$ for all prime p , then

$$X \cong (\Sigma_{\mathbf{a}})^n, \quad (5)$$

where $\mathbf{a} = (2, 3, 4, \dots)$, ([11, (25.8)]). It is obvious that it suffices to prove the lemma for the group of the form $X = \Sigma_{\mathbf{a}}, \mathbf{a} = (2, 3, 4, \dots)$. Then the group Y is topologically isomorphic to the group \mathbb{Q} . Let p and q be different prime numbers. Let H be a subgroup of Y of the form $H = \{\frac{m}{q^k}\}_{m, k \in \mathbb{Z}}$. Put $G = H^*, K = A(G, H^{(p)})$. Since numbers p and q are relatively prime, it follows that $H \neq H^{(p)}$. Consider on the group H the function

$$f(y) = \begin{cases} 1, & y \in H^{(p)}, \\ c, & y \notin H^{(p)}, \end{cases} \quad (6)$$

where $0 < c < 1$. By Lemma 3.7 $f(y)$ is a positive definite function.

for some prime number p . Suppose that p is the smallest one satisfying condition (11). Since X is a connected group, we have $X^{(n)} = X$ for all natural n . Hence if $f_p \notin \text{Aut}(X)$, then $\text{Ker} f_p \neq \{0\}$.

From the condition of the lemma it follows that $p \geq 3$. Put $a = 1 - p$. Since p is a smallest natural number satisfying condition (11), we obtain $f_{-a} \in \text{Aut}(X)$. Hence $f_a \in \text{Aut}(X)$. Note that $\text{Ker} f_p = A(X, Y^{(p)})$. It implies that $Y^{(p)} \neq Y$. Let $\tilde{y} \in Y^{(p)}$ and verify that the automorphism $\delta = f_a$ and the element \tilde{y} satisfy to conditions of Lemma 3.6. We have $\tilde{f}_a = f_a$ and $I - \tilde{f}_a = \tilde{f}_p$. Since Y is torsion-free group, it follows that $\text{Ker}(I - \tilde{f}_a) = \{0\}$, i.e. condition (i) holds. Thus $(I - \tilde{f}_a)Y = Y^{(p)}$. From $p \geq 3$ it follows that numbers 2 and p are relatively prime. Hence there are integers m and n such that $2m + pn = 1$. Thus $y = 2my + pny$. So if $\tilde{y} \notin Y^{(p)}$, then $2\tilde{y} \notin Y^{(p)}$ too. It implies that condition (ii) holds. Taking in the account that Y is torsion-free group, it is obvious that condition (iii) holds. We use Lemma 3.6 and obtain the assertion of the lemma. ■

4. Main theorem

Theorem 4.1. *Let $X = \Sigma_{\alpha}$ be an α -adic solenoid.*

1. *Assume that $f_p \notin \text{Aut}(X)$ for all prime numbers p . Let $\xi_i, i = 1, 2, 3$, be independent random variables with values in X and distributions μ_i . Then the independence of the linear forms $L_j = \sum_{i=1}^3 \alpha_{ij} \xi_i$, where $\alpha_{ij} \in \text{Aut}(X), i, j = 1, 2, 3$, implies that at least one distribution $\mu_i \in I(X)$.*

2. *Assume that $f_p \in \text{Aut}(X)$ for a prime number p . Then there are independent random variables $\xi_i, i = 1, 2, 3$, with values in X and distributions $\mu_i \notin \Gamma(X) * I(X)$, and automorphisms $\alpha_{ij} \in \text{Aut}(X)$, such that the linear forms $L_j = \sum_{i=1}^3 \alpha_{ij} \xi_i, j = 1, 2, 3$, are independent.*

It should be noted that an example of a group such that $f_p \notin \text{Aut}(X)$ for all prime number p is the group $\Sigma_{\alpha}, \alpha = (2, 3, 5, 7, \dots)$. Its character group $\Sigma_{\alpha}^* \cong \{\frac{m}{p_1 p_2 \dots p_k} : m \in \mathbb{Z}, p_j \text{ are different prime numbers}\}$.

An example of a group such that $f_p \in \text{Aut}(X)$ for a prime number p is the group $\Sigma_{\alpha}, \alpha = (2, 2, 2, \dots)$. Its character group $\Sigma_{\alpha}^* \cong \{\frac{m}{2^k} : m, k \in \mathbb{Z}\}$.

The proof of Theorem 4.1 is divided into two parts. In the first part we use Corollaries 3.3 and 3.5. In the second part we use Lemma 3.8.

Proof. 1. Suppose that $f_p \notin \text{Aut}(X)$ for all prime numbers p . This implies that $\text{Aut}(X) = \{I, -I\}$. It is easy to show that the case of arbitrary linear forms L_j is reduced to the case when L_j are of the form

$$\begin{aligned} L_1 &= \xi_1 + \xi_2 + \xi_3, \\ L_2 &= \xi_1 - \xi_2 + \xi_3, \\ L_3 &= \xi_1 - \xi_2 - \xi_3. \end{aligned} \tag{12}$$

Note that Y is topologically isomorphic to a subgroup of \mathbb{Q} . To avoid introducing new notation we will suppose that Y is a subgroup of \mathbb{Q} . By

Lemma 3.1 the independence of the linear forms (12) implies that equation (3.5), where Y is a subgroup of \mathbb{Q} , holds.

Note that, since $f_2 \notin \text{Aut}(X)$, we have that the partition of Y into the cosets of $Y^{(2)}$ consists of two cosets: $Y^{(2)}$ and $\tilde{y} + Y^{(2)}$, where $\tilde{y} \notin Y^{(2)}$.

Put $N_i = \{y \in Y : \hat{\mu}_i(y) \neq 0\}$, $N = \cap_{i=1}^3 N_i$. We infer from (3.5) that N is a subgroup in Y . Moreover, it is easy to see from (3.5), that N has a property:

$$\text{if } 2y \in N, \text{ then } y \in N. \quad (13)$$

There are two cases: $N \neq \{0\}$ and $N = \{0\}$.

A. Assume that $N \neq \{0\}$. Suppose that t_1 and t_2 belong to the same coset of $Y^{(2)}$ in Y . Then there exist \hat{u}_1 and \hat{u}_2 , such that $\hat{u}_1 + \hat{u}_2 = t_1$, $\hat{u}_1 - \hat{u}_2 = t_2$. Putting first $u_1 = \hat{u}_1$, $u_2 = \hat{u}_2$, $u_3 = 0$ in (3.5), then $u_1 = \hat{u}_1$, $u_2 = -\hat{u}_2$, $u_3 = 0$ in (3.5), and equating the right-hand sides of obtained equations, we infer:

$$|\hat{\mu}_1(t_1)| |\hat{\mu}_2(t_2)| |\hat{\mu}_3(t_1)| = |\hat{\mu}_1(t_2)| |\hat{\mu}_2(t_1)| |\hat{\mu}_3(t_2)|.$$

Reasoning the same way, it is easy to see that if t_1 and t_2 belong to the same coset of $Y^{(2)}$ in Y , then the following equation holds:

$$|\hat{\mu}_{i_1}(t_1)| |\hat{\mu}_{i_2}(t_2)| |\hat{\mu}_{i_3}(t_2)| = |\hat{\mu}_{i_1}(t_2)| |\hat{\mu}_{i_2}(t_1)| |\hat{\mu}_{i_3}(t_1)|, \quad (14)$$

where all i_j are pairwise different.

Put $\nu_i = \mu_i * \bar{\mu}_i$, $i = 1, 2, \dots, n$. Then $\hat{\nu}_i(y) = |\hat{\mu}_i(y)|^2$, $y \in Y$. Functions $\hat{\nu}_i(y)$ are nonnegative and also satisfy equation (3.5). It suffices to show that $\hat{\nu}_i(y)$ are characteristic functions of the idempotent distributions. This implies that $\hat{\mu}_i(y)$ are also characteristic functions of the idempotent distributions.

Now we will show that $N_i = N$, $i = 1, 2, 3$. Assume the converse. Then there exists $y_1 \in N_{i_1}$ such that either $y_1 \notin N_{i_2}$ or $y_1 \notin N_{i_3}$, where all i_j are pairwise different. Put $t_1 = y_1$, $t_2 = y_2$, where $y_2 \in N$ and y_1, y_2 belong to the same coset of the $Y^{(2)}$ in Y , in (14). We can make such choice. Indeed, on the one hand $N \cap Y^{(2)} \neq \{0\}$ because N is a subgroup and $N \neq \{0\}$ by the assumption. On the other hand there exists $y \neq 0$ such that $y \in N \cap (\tilde{y} + Y^{(2)})$. Indeed, if $N \subset Y^{(2)}$, then taking in the account (13) we infer that there exists $y' \in Y$ such that $y = 2^k y'$. This contradicts to the fact that there are no $y \in Y$ such that y is infinitely divisible by 2. We infer that the left-hand side of equation (14) is equal to a positive number, and the right-hand side of equation (14) is equal to zero. This is a contradiction. So we have that $N_i = N$, $i = 1, 2, 3$.

Note that if $y \in N$, then $\hat{\nu}_i(y) = 1$, $i = 1, 2, 3$. Indeed, let $y_0 \in N$. Consider the subgroup H of Y generated by y_0 . Note that $H \cong \mathbb{Z}$. Consider the restriction of equation (3.5) to the subgroup H . Using Corollary 3.5 we obtain that $\hat{\nu}_i(y) = 1$, $i = 1, 2, 3$, $y \in H$.

Taking in the account that the characteristic functions $\hat{\nu}_i(y)$ are N -invariant, consider the equation induced by equation (3.5) on the factor-group Y/N . Put $f_i([y]) = \hat{\nu}_i([y])$. Note that if H is an arbitrary nontrivial subgroup of Y , then Y/H is topologically isomorphic to a group of the form $\mathbf{P}_{p \in \mathcal{P}}^* \mathbb{Z}(p^{k_p})$, where $k_p \geq 0$. In particular, this holds for the factor-group

Y/N . Hence, by Corollary 3.3 we infer that $f_i([y])$ are characteristic functions of some idempotent distributions. This implies that all distributions μ_i are idempotent.

B. Consider the case $N = \{0\}$.

Put first $u_2 = 0, u_3 = u_1 = y$, after $u_3 = 0, u_1 = u_2 = y$, and finally $u_1 = 0, u_2 = u_3 = y$ in (3.5), we infer respectively:

$$\hat{\mu}_1(2y) = \hat{\mu}_1^2(y) |\hat{\mu}_2(y)|^2 |\hat{\mu}_3(y)|^2, \quad y \in Y. \quad (15)$$

$$\hat{\mu}_2(2y) = |\hat{\mu}_1(y)|^2 \hat{\mu}_2^2(y) |\hat{\mu}_3(y)|^2, \quad y \in Y. \quad (16)$$

$$\hat{\mu}_3(2y) = |\hat{\mu}_1(y)|^2 |\hat{\mu}_2(y)|^2 \hat{\mu}_3^2(y), \quad y \in Y. \quad (17)$$

Note that

$$\hat{\mu}_i(2y) = 0, y \in Y, y \neq 0, i = 1, 2, 3. \quad (18)$$

Indeed, if $\hat{\mu}_{i_0}(2y_0) \neq 0$ for some $y_0 \in Y, y_0 \neq 0$, and i_0 , then from equalities (15)-(17) it follows that $\hat{\mu}_i(y_0) \neq 0, i = 1, 2, 3$. This contradicts to $N \neq \{0\}$.

Show that at least one distribution $\mu_i = m_X$. Assume the converse. Then there exist $t_1 \neq 0, t_2 \neq 0, t_3 \neq 0$, such that

$$\hat{\mu}_1(\pm t_1) \hat{\mu}_2(\pm t_2) \hat{\mu}_3(\pm t_3) \neq 0. \quad (19)$$

From equality (18) it follows that $t_i \in \tilde{y} + Y^{(2)}$. From $N = \{0\}$ it follows that $\pm t_i, i = 1, 2, 3$, do not coincide. Without loss of generality assume that $t_1 \neq \pm t_2$. Note that for all elements $y', y'' \in \tilde{y} + Y^{(2)}$ we have $y' + y'' \in Y^{(2)}$. Moreover, for any two elements $y', y'' \in \tilde{y} + Y^{(2)}$ there are two possibilities: either $y' + y'' \in Y^{(4)}$ or $y' - y'' \in Y^{(4)}$.

Put $y_i = t_i, i = 1, 2, 3$, if $t_1 + t_2 \in Y^{(4)}$, and put $y_1 = t_1, y_2 = -t_2, y_3 = t_3$, if $t_1 - t_2 \in Y^{(4)}$. Note that $y_1 + y_2 \in Y^{(4)}, y_1 + y_2 \neq 0$. For an element $y_0 \in Y^{(2)}$ denote by $\frac{y_0}{2}$ such element of Y , that $2\frac{y_0}{2} = y_0$. Thus we infer that $\frac{y_1 + y_2}{2} \in Y^{(2)}, \frac{y_1 + y_2}{2} \neq 0$.

Consider the system of equations

$$\begin{cases} u_1 + u_2 + u_3 = y_1, \\ u_1 - u_2 - u_3 = y_2, \\ u_1 + u_2 - u_3 = y_3. \end{cases} \quad (20)$$

Taking in the account that $y_i \in \tilde{y} + Y^{(2)}, i = 1, 2, 3$, it is easy to see that the system of equations (20) has the following solutions

$$\begin{cases} u_1 = \frac{y_1 + y_2}{2}, \\ u_2 = \frac{y_3 - y_2}{2}, \\ u_3 = \frac{y_1 - y_3}{2} \end{cases} \quad (21)$$

Put the solutions of (21) in equation (3.5). Taking into the account (19) we infer that the right-hand side of (3.5) is not equal to 0. This implies that

$$\hat{\mu}_1\left(\frac{y_1 + y_2}{2}\right) \hat{\mu}_2\left(\frac{y_3 - y_2}{2}\right) \hat{\mu}_3\left(\frac{y_1 - y_3}{2}\right) \neq 0. \quad (22)$$

It follows from inequality (22) that $\mu_1(\frac{y_1+y_2}{2}) \neq 0$. However, we have $\frac{y_1+y_2}{2} \in Y^{(2)}$. This contradicts to (18).

Note that we proved more: if $N \neq \{0\}$ then all distributions μ_i are idempotent, and if $N = \{0\}$ at least one distribution μ_i is the Haar distribution on X .

2. Now consider the case $f_p \in \text{Aut}(X)$ for some prime p . If $f_2 \in \text{Aut}(X)$, then the statement follows from the Lemma 3.8. Assume that $f_2 \notin \text{Aut}(X)$. Then two cases are possible: $p-1 = 4k$ and $p+1 = 4k$. Let us study the first case.

Consider the function $\rho(x)$ on X defined by the equation

$$\rho(x) = 1 + \text{Re}(x, y_0),$$

where $y_0 \in Y, y_0 \neq 0$. It is obvious that $\rho(x) \geq 0, x \in X$, and $\int_X \rho(x) dm_X(x) = 1$. Let μ be a distribution on X with the density $\rho(x)$ with respect to m_X . It is obvious that $\mu \notin \Gamma(X) * I(X)$. The characteristic function of the distribution μ is of the form:

$$\hat{\mu}(y) = \begin{cases} 1, & y = 0, \\ \frac{1}{2}, & y = \pm y_0, \\ 0, & y \notin \{0, y_0, -y_0\}. \end{cases} \quad (23)$$

Let $\xi_i, i = 1, 2, 3$, be independent identically distributed random variables with values in X and distribution μ . Let us verify that the linear forms $L_1 = \xi_1 + \xi_2 + \xi_3, L_2 = \xi_1 + p\xi_2 + \xi_3, L_3 = \xi_1 + \xi_2 + p\xi_3$ are independent. By Lemma 3.1 it suffices to prove that $\hat{\mu}(y)$ satisfies equation (2), which takes the form

$$\hat{\mu}(u+v+t)\hat{\mu}(u+pv+t)\hat{\mu}(u+v+pt) = \hat{\mu}^3(u)\hat{\mu}^2(v)\hat{\mu}^2(t)\hat{\mu}(pv)\hat{\mu}(pt), \quad (24)$$

where $u, v, t \in Y$. We will show that equation (24) holds. It is obvious, that it suffices to consider the case, when at least two of three elements u, v, t are not equal to 0. It is easy to see that in this case the right-hand side of equation (24) is equal to 0. Let us show that the left-hand side of equation (24) vanishes too.

Suppose that there are some elements u, v, t such that the left-hand side of equation (24) does not vanish. Then there exist some $h_i \in \{0, y_0, -y_0\}, i = 1, 2, 3$, such that u, v, t satisfy the system of equations

$$\begin{cases} u + v + t = h_1, \\ u + pv + t = h_2, \\ u + v + pt = h_3. \end{cases} \quad (25)$$

It is easy to obtain from (25) that

$$(p-1)v, (p-1)t \in \{0, \pm y_0, \pm 2y_0\}. \quad (26)$$

Relationship (26) fails because of $(p-1) = 4k$, but $y_0 \notin Y^{(2)}$. From this it follows that the left-hand side of equation (24) is equal to 0.

The second case can be studied similarly. In this case we have to consider the linear forms $L_1 = \xi_1 + \xi_2 + \xi_3, L_2 = \xi_1 - p\xi_2 + \xi_3, L_3 = \xi_1 + \xi_2 - p\xi_3$.

The theorem is completely proved.



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